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Quadrature-Free Characteristic Methods for Convection-Diffusion Problems

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Abstract

Galerkin-characteristics finite element method is a powerful numerical procedure for convection-diffusion problems even in the convection-dominated case. It is, however, reported that rough numerical integration for composite terms ruins the convergence property. In this paper we discuss two ways to avoid numerical quadrature referring to the recent results. One way is to use the lumping technique. A Galerkin-characteristics finite element scheme of lumped mass type is considered. The other way is to use a finite difference method derived from a Galerkin-characteristics finite element scheme. For these schemes the stability and convergence are discussed.

Keywords: convection-diffusion, characteristics, quadrature-free, finite element method, finite difference method, lumped mass, Péclet number, stability, convergence.

1 Introduction

Convection-diffusion problems appear and are solved in various fields of sciences and technologies. The convection-diffusion equation is linear, but to solve it is not always an easy task. When the Péclet number is high, that is, convection dominant cases, it is well-known that the Galerkin finite element scheme, or equivalently, the centered finite difference scheme, produces easily oscillation solutions. Hence, elaborate numerical schemes with new ideas have been developed to perform stable computation. Among them we focus on the method based on characteristics [1].

The procedure of the characteristic method is natural from the physical point of view since it approximates particle movements. It is also attractive from the computational point of view since it leads to a symmetric system of linear equations. Schemes derived from the characteristic method are recognized to be robust for high-Péclet numbers. Galerkin-characteristics method also has the advantage of the finite element method, the geometrical flexibility and the extension to higher-order schemes. A unique disadvantage of this method is in the computation of composite function terms. Since the terms are not polynomials, some numerical quadrature is usually employed to compute them. It is, however, reported that much attention should be paid to the numerical quadrature, because a rough numerical integration formula may yield oscillating results caused by the non-smoothness of the composite function terms [2].

In this paper we discuss two ways to avoid numerical quadrature, referring to the recent results. One way is to use the lumping technique. A Galerkin-characteristics finite element scheme of lumped mass type [3] is considered. The other way is to use a finite difference method[4] derived from a Galerkin-characteristics finite element scheme[5]. Both schemes are free from numerical quadrature. For these schemes the stability and convergence are discussed.

Throught the paper the symbol c with or without subscripts is used for a generic positive constant independent of the discretization parameters, which may take a different value at each occurrence. We often write $C^j(X)$ in place of $C^j([0,T];X)$ if there is no confusion.

2 Basic idea of the method of characteristics

Let Ω be a bounded domain in $\mathbb{R}^2$ and T be a positive constant. We consider an initial boundary value problem; find $\phi : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla\phi - \nu\Delta\phi = f \qquad \text{in } \Omega \times (0,T), \tag{1a}$$

$$\phi = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1b}$$

$$\phi(\cdot, 0) = \phi^0 \qquad \text{in } \Omega, \tag{1c}$$

where ν is a positive constant, $u \in C(\bar{\Omega} \times [0,T]; \mathbb{R}^2)$ and $f \in C(\bar{\Omega} \times [0,T]; \mathbb{R})$ are given functions. Physically u stands for the velocity of a fluid. We assume that it vanishes on the boundary.

Hypothesis 1. *The velocity u satisfies*

$$u \in C\left([0,T]; W^{1,\infty}(\Omega)\right), \quad u = 0 \ \ (x \in \partial\Omega).$$

We denote the material derivative by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla.$$

Let ψ be any function in $V \equiv H_0^1(\Omega)$. The weak formulation of (1) is

$$\left(\frac{D\phi}{Dt}(t), \psi\right) + \nu(\nabla\phi(t), \nabla\psi) = (f(t), \psi), \quad (t \in (0,T)), \tag{2}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

Let $X\colon (0,T) \to \mathbb{R}^2$ be a solution of the ordinary differential equation

$$\frac{dX}{dt} = u(X,t). \tag{3}$$

Then, we can write

$$\frac{D\phi}{Dt}(X(t),t) = \frac{d}{dt}\phi(X(t),t). \tag{4}$$

Let Δt be a time increment. We set $N_T = \lfloor T/\Delta t \rfloor$ and $t^n = n\Delta t$ for $n \in \mathbb{Z}$. Let (x, t^n) be a given point in $\Omega \times (0,T)$. We denote by $X(t; t^n, x)$ the solution of (3) subject to an initial condition $X(t^n) = x$. By the backward Euler approximation the left-hand side of (4) is approximated by

$$\frac{D\phi}{Dt}(x,t^n) \approx \frac{\phi(x,t^n) - \phi(X(t^{n-1}), t^{n-1})}{\Delta t},$$

which is the basic idea of the method of characteristics used in the numerical analysis of flow problems. It approximates the material derivative term along the particle path $(X(t), t)$. Let V_h be a finite element space in V. We denote by $\phi_h^n \in V_h$ an approximation to $\phi(\cdot, n\Delta t) \in V$. The Galerkin-characteristics finite element scheme for (1) is to find $\{\phi_h^n;\ n = 1, \cdots, N_T\} \subset V_h$ such that

$$\left(\frac{\phi_h^n - \phi_h^{n-1} \circ X_1^n}{\Delta t}, \psi_h\right) + \nu(\nabla\phi_h^n, \nabla\psi_h) = (f^n, \psi_h) \quad (\forall \psi_h \in V_h) \tag{5}$$

subject to an initial condition $\phi_h^0 \in V_h$, where

$$X_1^n(x) = x - u^n(x)\Delta t$$

and $\circ$ means the composition of functions, $(\phi_h^{n-1} \circ X_1^n)(x) = \phi_h^{n-1}(X_1^n(x))$. Since $X_1^n(x)$ approximates $X(t^{n-1}; t^n, x)$, the first term of the left-hand side of (5) is an approximation of the first term of (2).

The Galerkin-characteristics finite element scheme has advantages as follows [1, 5]:

- geometrical flexibility since it is a finite element method;
- robustness for high Péclet number problems; it works even for pure convection problems;
- unconditonal stability; there is no restriction on the choice of Δt;
- symmetricity of the matrix; the resultant matrix to be solved at each time step is symmetric and independent of the step number.

The scheme (5) is of the first order in Δt. The second order Galerkin-characteristics scheme in Δt of Crank-Nicolson type is given by [5] as

$$\begin{aligned}\left(\frac{\phi_h^n-\phi_h^{n-1}\circ X_2^n}{\Delta t},\psi_h\right)&+\frac{\nu}{2}\left(\nabla\phi_h^n+\nabla\phi_h^{n-1}\circ X_1^n,\nabla\psi_h\right)\\&+\frac{\nu\Delta t}{2}\left\{\left(J^n\nabla\phi_h^{n-1},\nabla\psi_h\right)+\left(\nabla(\nabla\cdot u^n)\cdot\nabla\phi_h^{n-1},\psi_h\right)\right\}\\&=\frac{1}{2}(f^n+f^{n-1}\circ X_1^n,\psi_h)\quad(\forall\psi_h\in V_h),\end{aligned}\tag{6}$$

where J^n is the Jacobi matrix defined by

$$J_{ij}^n=\frac{\partial u_i^n}{\partial x_j}$$

and X_2^n is the approximation of $X(t^{n-1};t^n,x)$ obtained by the second order Runge-Kutta method or the Heun method,

$$\begin{aligned}X_2^n(x)&=x-u^{n-1/2}\left(x-u^n(x)\frac{\Delta t}{2}\right)\Delta t,\\X_2^n(x)&=x-\Big(u^n(x)+u^{n-1}(x-u^n(x)\Delta t)\Big)\frac{\Delta t}{2}.\end{aligned}$$

This scheme has also all the advantages mentioned above. For these schemes the convergence of the finite element solution ϕ_h to the exact one ϕ is proved to be of $O(\Delta t+h^k)$ and $O(\Delta t^2+h^k)$, respectively, in a norm corresponding to $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ when P_k finite element is employed [5].

In these schemes we have to compute the term such as

$$\int_K \phi_h^{n-1}\circ X_k^n\,w_{hi}\,dx\quad(k=1,2),$$

where K is an element and w_{hi} is the base function associated with node P_i. Since the composite function $\phi_h^{n-1}\circ X_k^n$ is neither a polynomial nor smooth on K, some numerical integration is often employed. However, it is reported in [6] that rough numerical integration causes oscillation and that even the interruption of the computation is encountered by overflow in the worst case though the unconditional stability is proved. That is, quadrature error ruins the stability property of the schemes. This inconvenience is improved if we use the second order scheme (6) as shown in [2], but much attention should be paid in the treatment of these terms.

We focus on schemes based on the method of characteristics and they are free of numerical integration. One way of constructing such a scheme is the finite difference method and the other one is the lumping.

3 A finite difference scheme of the first order in time

Assume that Ω is a rectangle, $[0,L_1]\times[0,L_2]$. Let h be a lattice size, and we assume that $N_1=L_1/h$ and $N_2=L_2/h$ be integers. We denote by $x_{i,j}$ the lattice point

(ih, jh). We prepare sets of lattice points,

$$
\begin{aligned}
\bar{\Omega}_h &= \{x_{i,j};\ i = 0, \cdots, N_1,\ j = 0, \cdots, N_2\}, \\
\Omega_h &= \{x_{i,j};\ i = 1, \cdots, N_1 - 1,\ j = 1, \cdots, N_2 - 1\}, \\
\Gamma_h &= \bar{\Omega}_h \backslash \Omega_h.
\end{aligned}
$$

Let ϕ_h be a lattice function defined on $\bar{\Omega}_h$. We set $\phi_{h\,i,j} = \phi_h(x_{i,j})$ for $x_{i,j} \in \bar{\Omega}_h$. We define the bilinear interpolation operator $\Pi_h^{(1)}$ into $C(\bar{\Omega})$ by

$$
\begin{aligned}
(\Pi_h^{(1)}\phi_h)(x) =& \phi_{h\,i,j}\, w_{i,j}(x) + \phi_{h\,i+1,j}\, w_{i+1,j}(x) + \phi_{h\,i+1,j+1}\, w_{i+1,j+1}(x) \\
& + \phi_{h\,i,j+1}\, w_{i,j+1}(x) \qquad \big(x \in [ih, ih+h) \times [jh, jh+h)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
i &= \lfloor \frac{x_1}{h} \rfloor, \quad j = \lfloor \frac{x_2}{h} \rfloor, \\
w_{i,j}(x) &= \left(i + 1 - \frac{x_1}{h}\right)\left(j + 1 - \frac{x_2}{h}\right), \quad w_{i+1,j}(x) = \left(\frac{x_1}{h} - i\right)\left(j + 1 - \frac{x_2}{h}\right) \\
w_{i,j+1}(x) &= \left(i + 1 - \frac{x_1}{h}\right)\left(\frac{x_2}{h} - j\right), \qquad w_{i+1,j+1}(x) = \left(\frac{x_1}{h} - i\right)\left(\frac{x_2}{h} - j\right).
\end{aligned}
$$

We denote by V_h a set of lattice functions defined by

$$
V_h = \left\{v_h : \bar{\Omega}_h \to \mathbf{R};\ v_h = 0 \text{ on } \Gamma_h\right\}.
$$

A characteristics finite difference scheme for (1) is to find $\{\phi_h^n;\ n = 1, \cdots N_T\} \subset V_h$ such that

$$
\frac{\phi_h^n - (\Pi_h^{(1)}\phi_h^{n-1}) \circ X_1^n}{\Delta t} - \nu \Delta_h \phi_h^n = f^n \quad \text{ on } \Omega_h \tag{7}
$$

subject to $\phi_h^0 = \phi^0$ on $\bar{\Omega}_h$, where Δ_h is the discrete Laplace operator defined by

$$
(\Delta_h \phi_h)(x_{i,j}) = \frac{1}{h^2}\left(\phi_{h\,i+1,j} + \phi_{h\,i-1,j} + \phi_{h\,i,j+1} + \phi_{h\,i,j-1} - 4\,\phi_{h\,i,j}\right).
$$

Remark 1. *In general, $X_1^n(x_{i,j}) = x_{i,j} - u^n(x_{i,j})\Delta t$ is not a lattice point. Therefore, the interpolation operator $\Pi_h^{(1)}$ is necessary in* (7)*. The equation* (7) *can be written equivalently as*

$$
\begin{aligned}
&(1 + \frac{4\nu\Delta t}{h^2})\phi_{h\,i,j}^n - \frac{\nu\Delta t}{h^2}\phi_{h\,i+1,j}^n - \frac{\nu\Delta t}{h^2}\phi_{h\,i-1,j}^n - \frac{\nu\Delta t}{h^2}\phi_{h\,i,j+1}^n - \frac{\nu\Delta t}{h^2}\phi_{h\,i,j-1}^n \\
&= w_{k,\ell}(y_{i,j})\phi_{h\,k,\ell}^{n-1} + w_{k+1,\ell}(y_{i,j})\phi_{h\,k+1,\ell}^{n-1} + w_{k+1,\ell+1}(y_{i,j})\phi_{h\,k+1,\ell+1}^{n-1} \\
&\quad + w_{k,\ell+1}(y_{i,j})\phi_{h\,k,\ell+1}^{n-1} + \Delta t\, f_{i,j}^n,
\end{aligned} \tag{8}
$$

where

$$
y_{i,j} = x_{i,j} - u^n(x_{i,j})\Delta t, \quad k = \lfloor \frac{(y_{i,j})_1}{h} \rfloor, \quad \ell = \lfloor \frac{(y_{i,j})_2}{h} \rfloor.
$$

Since (7) is a finite difference scheme, no quadrature is required. We show the stability and convergence results. We use the following norms,

$$\|\phi_h\|_{\ell^\infty(\ell^\infty)} = \max_{0\le n\le N_T} \|\phi_h^n\|_{\ell^\infty}, \quad \|\phi_h\|_{\ell^1(\ell^\infty)} = \Delta t \sum_{n=0}^{N_T} \|\phi_h^n\|_{\ell^\infty},$$
$$\|\phi_h^n\|_{\ell^\infty} = \max\{|\phi_h^n(x_{i,j})|;\ x_{i,j} \in \bar\Omega_h\}.$$

Hypothesis 2.

$$\Delta t < \frac{1}{\|u\|_{C(W^{1,\infty}(\Omega))}}. \tag{9}$$

Lemma 1. *Under Hypotheses 1 and 2, scheme* (7) *is L^∞-stable, i.e., it holds*

$$\|\phi_h\|_{\ell^\infty(\ell^\infty)} \le \|\phi_h^0\|_{\ell^\infty} + \|f\|_{\ell^1(\ell^\infty)}. \tag{10}$$

Proof. Condition (9) implies that $X_1^n(x) \in \bar\Omega$ as shown in [5, Proposition 1] . In general, it holds that for any $y \in \bar\Omega$

$$w_{k,\ell}(y),\ w_{k+1,\ell}(y),\ w_{k+1,\ell+1}(y),\ w_{k,\ell+1}(y) \ge 0,$$
$$w_{k,\ell}(y) + w_{k+1,\ell}(y) + w_{k+1,\ell+1}(y) + w_{k,\ell+1}(y) = 1 \quad \left(k = \lfloor \frac{y_1}{h} \rfloor,\ \ell = \lfloor \frac{y_2}{h} \rfloor\right).$$

Hence we can derive the result from (8) by a maximum principle. □

We introduce the function space

$$Z_C^m = \{\phi \in C^j(C^{m-j}(\bar\Omega));\ j = 0, \cdots, m, \|\phi\|_{Z_C^m} < +\infty\},$$
$$\|\phi\|_{Z_C^m} \equiv \max\{\|\phi\|_{C^j(C^{m-j}(\bar\Omega))};\ j = 0, \cdots, m\}.$$

Theorem 1. *Assume hypotheses 1 and 2. Let ϕ and ϕ_h be the solution of* (1) *and* (7)*, respectively. If ϕ and u satisfy*

$$\phi \in Z_C^2 \cap C^0([0,T]; C^3(\bar\Omega)), \quad u \in Z_C^1, \tag{11}$$

then we have

$$\|\phi_h - \phi\|_{\ell^\infty(\ell^\infty)} \le c(\|u\|_{Z_C^1})\|\phi\|_{Z_C^2 \cap C(C^3)}(h + \Delta t).$$

Proof. Let $e_h^n = \phi_h^n - \phi^n$ be the function defined on $\bar\Omega_h$. Then $e_h = \{e_h^n; n = 1, \cdots, N_T\}$ satisfies (7) with the initial condition $e_h^0 = 0$ and the right-hand side R^n defined by

$$R^n \equiv R_1^n + R_2^n + R_3^n + R_4^n,$$
$$R_1^n \equiv \frac{D\phi^n}{Dt} - \frac{\phi^n - \phi^{n-1}\circ X^{n-1}}{\Delta t},$$
$$R_2^n \equiv \frac{\phi^{n-1}\circ X_1^n - \phi^{n-1}\circ X^{n-1}}{\Delta t},$$
$$R_3^n \equiv \frac{\Pi^{(1)}\phi^{n-1}\circ X_1^n - \phi^{n-1}\circ X_1^n}{\Delta t},$$
$$R_4^n \equiv \nu\Delta_h\phi^n - \nu\Delta\phi^n.$$

Let $g(t) = \phi(X(t;t^n,x),t)$. Then, we have

$$
\begin{aligned}
|R_1^n| &= |\frac{dg}{dt}(t^n) - \frac{g(t^n) - g(t^{n-1})}{\Delta t}| \\
&\le \frac{1}{2}\|\frac{d^2 g}{dt}\|_{L^\infty(t^{n-1},t^n)}\Delta t \le c(\|u\|_{Z_C^1})\|\phi\|_{Z_C^2}\Delta t.
\end{aligned}
$$

The second terms is evaluated as

$$
\begin{aligned}
|R_2^n| &= |\frac{1}{\Delta t}\nabla\phi^{n-1}(X_1^n - X^{n-1})| \\
&\le c(\|u\|_{Z_C^1})\|\phi\|_{C(C^1)}\Delta t.
\end{aligned}
$$

Noting that

$$
(\Pi^{(1)} - I)\phi^{n-1} = 0 \quad (\forall x_{i,j} \in \Omega_h),
$$

we evaluate the third term as

$$
\begin{aligned}
|R_3^n| &= |\frac{1}{\Delta t}(\Pi^{(1)} - I)\phi^{n-1} \circ X_1^n| \\
&= |\frac{1}{\Delta t}\{(\Pi^{(1)} - I)\phi^{n-1} \circ X_1^n - (\Pi^{(1)} - I)\phi^{n-1} \circ I\}| \\
&= \frac{1}{\Delta t}|\nabla\{(\Pi^{(1)} - I)\phi^{n-1}\}|\,|X_1^n - x| \\
&\le c(\|u\|_C)|\nabla\{(\Pi^{(1)} - I)\phi^{n-1}\}| \\
&\le c(\|u\|_C)\|\phi\|_{C(C^2)}h.
\end{aligned}
$$

The fourth term is evaluated as

$$
|R_4^n| \le c\nu\|\phi\|_{C(C^3)}h.
$$

Gathering these estimates, we obtain

$$
|R^n| \le c(\|u\|_{Z_C^1})\|\phi\|_{Z_C^2 \cap C(C^3)}(h + \Delta t),
$$

which implies the result by virtue of Lemma 1. □

4 A Galerkin-characteristics finite element scheme of lumped mass type

The result of the previous section can be extended to a finite element method of lumped mass type [3]. Lumped mass technique requires no quadrature.

Let $\mathcal{T}_h \equiv \{K\}$ be a partition of Ω by triangles. Let $X_h(\subset H^1(\Omega))$ be the P1-finite element space, and V_h be $X_h \cap H_0^1(\Omega)$. A Galerkin-characteristics finite element scheme of lumped mass is to find $\{\phi_h^n;\ n = 1, \cdots, N_T\} \subset V_h$ such that

$$
\left(\frac{\bar\phi_h^n - \bar I_h(\phi_h^{n-1} \circ X_1^n)}{\Delta t}, \bar\psi_h\right) + \nu(\nabla\phi_h^n, \nabla\psi_h) = \left(\bar I_h f^n, \bar\psi_h\right), \quad \forall\psi_h \in V_h \tag{12}
$$

subject to the initial condition $\phi_h^0 = I_h \phi^0$. Here, $I_h : C(\bar{\Omega}) \to X_h$ is the interpolation operator defined by

$$(I_h \psi)(P) = \psi(P), \quad (\forall \text{node } P \in \bar{\Omega}),$$

$\bar{}: V_h \to L^2(\Omega)$ is the lumping operator defined by

$$\bar{\psi}_h(x) = \psi_h(P), \quad (x \in D_P)$$

and D_P is the barycentric domain [7] associated with node P shown in Fig. 1,

$$D_P = \bigcup_K \{D_P^K; P \in K \in \mathcal{T}_h\}$$
$$D_P^K = \bigcap_{j=1}^{2} \{x; x \in K, \lambda_{Q(j)}(x) \leq \lambda_P(x)\},$$

where $\{P, Q(1), Q(2)\}$ is the set of the vertices of K and $\{\lambda_P,\ \lambda_{Q(1)}, \lambda_{Q(2)}\}$ is the system of the barycentric coordinates.

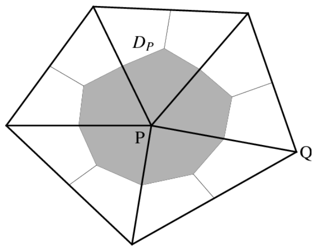

Figure 1: The barycentric domain D_P associted with P.

Let N be the number of interior nodes, and w_{hi}, $i = 1, \cdots, N$, be the base function associated with node $P_i \in \Omega$,

$$w_{hi} \in V_h, \quad w_{hi}(P_j) = \delta_{ij},\ i, j = 1, \cdots, N.$$

Let $A = \{a_{ij}\}$ be the stiffness matrix with

$$a_{ij} = (\nabla w_{hj}, \nabla w_{hi}), \quad i, j = 1, \cdots, N.$$

Hypothesis 3. $a_{ij} \leq 0$ *for* $i \neq j,\ i, j = 1, \cdots, N.$

Remark 2. *A sufficient condition for Hypothesis 3 is that for any edge E it holds that*

$$\alpha_1^E + \alpha_2^E \leq \pi,$$

where α_i^E, $i = 1, 2$, are two angles (of two elements sharing E) opposite to E [8]. A little stronger but more familiar condition is every angle of all triangles is less than or equal to $\pi/2$ [9].

Let m_i be

$$m_i = \text{meas}\, D_{P_i}.$$

Setting $\psi_h = w_{hi}$ in (12) and dividing the i-th equation by m_i, we obtain an equivalent equation to (12), for $i = 1, \cdots, N$,

$$\frac{1}{\Delta t}\left(\phi_h^n(P_i) - (\phi_h^{n-1} \circ X_1^n)(P_i)\right) + \nu \frac{1}{m_i} \sum_{j=1}^{N} a_{ij} \phi_h^n(P_j) = f^n(P_i). \tag{13}$$

For a set of functions $\{\phi^n\}_{n=0}^{N_T} \subset L^\infty(\Omega)$ we define norms,

$$\|\phi\|_{\ell^\infty(L^\infty)} = \max_{0 \le n \le N_T} \|\phi^n\|_{L^\infty(\Omega)}, \quad \|\phi\|_{\ell^1(L^\infty)} = \Delta t \sum_{n=0}^{N_T} \|\phi^n\|_{L^\infty(\Omega)}. \tag{14}$$

Lemma 2. *Under Hypotheses 1, 2 and 3, scheme* (12) *is L^∞-stable, i.e., it holds*

$$||\phi_h||_{\ell^\infty(L^\infty)} \le ||\phi_h^0||_{L^\infty} + ||I_h f||_{\ell^1(L^\infty)}. \tag{15}$$

The proof is similar to that of Theorem 1.

Theorem 2. *Assume hypotheses 1, 2 and 3. Let ϕ and ϕ_h be the solution of* (1) *and* (12)*, respectively. If ϕ and u satisfy* (11)*, then for any $\epsilon \in (0,1)$ we have*

$$||\phi_h - I_h\phi||_{\ell^\infty(L^\infty)} \le c_\epsilon(\|u\|_{Z_C^1})\|\phi\|_{Z_C^2 \cap C(C^3)} \left(h + \Delta t + \frac{h^{2-\epsilon}}{\Delta t}\right). \tag{16}$$

By taking $\Delta t = O(h)$ we have

$$||\phi_h - I_h\phi||_{\ell^\infty(L^\infty)} \le c_\epsilon(\|u\|_{Z_C^1})\|\phi\|_{Z_C^2 \cap C(C^3)} h^{1-\epsilon}. \tag{17}$$

We omit the proof, which is found in [3].

Remark 3. *The ϵ of the power in h is the consequence of the error estimate for the Poisson equation by the P1 finite element [9]. It disappears for three dimensional problems.*

5 A characteristics finite difference scheme of second order in time

We present a characteristics finite difference scheme of second order in time [4], which corresponds to (6). Since it is a finite difference scheme, no quadrature is required.

Here we consider again functions ϕ_h defined on the lattice set $\bar{\Omega}_h$. A finite difference scheme corresponding to (6) is

$$
\begin{aligned}
&\frac{\phi_h^n - \big(\Pi_h^{(1)}\phi_h^{n-1}\big)\circ X_2^n}{\Delta t} - \frac{\nu}{2}\Big(\Delta_h\phi_h^n + \tilde{\Delta}_h^{(n)}\phi_h^{n-1}\Big) \\
&\quad - \frac{\nu\Delta t}{2}\Big\{\frac{\partial u_1^n}{\partial x_1}\nabla_{h1}\nabla_{h1} + \frac{\partial u_2^n}{\partial x_2}\nabla_{h2}\nabla_{h2} + (\frac{\partial u_1^n}{\partial x_2} + \frac{\partial u_2^n}{\partial x_1})\nabla_{(2h)1}\nabla_{(2h)2}\Big\}\phi_h^{n-1} \\
&= \frac{1}{2}(f^n + f^{n-1}\circ X_1^n) \quad \text{on } \Omega_h,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
\tilde{\Delta}_h^{(n)} &\equiv \nabla_{h1}\tilde{\nabla}_{h1}^{(n)} + \nabla_{h2}\tilde{\nabla}_{h2}^{(n)}, \\
(\nabla_{hk}\, v_h)(x) &\equiv \frac{1}{h}\big\{v_h(x+\frac{h}{2}e_k) - v_h(x-\frac{h}{2}e_k)\big\}, \\
\tilde{\nabla}_{hk}^{(n)} &\equiv \big\{\Pi_h^{\frac{1}{2}e_k,(1)}\big(\nabla_{hk}\,\cdot\big)\big\}\circ X_1^n, \\
(\nabla_{(2h)k}\, v_h)(x) &\equiv \frac{1}{2h}\big\{v_h(x+he_k) - v_h(x-he_k)\big\} \qquad (k=1,2),
\end{aligned}
$$

e_k is the unit vector to x_k direction, and $\Pi_h^{\frac{1}{2}e_k,(1)}$ is the bilinear interpolation operators using the values at $x_{i,j} \pm \frac{1}{2}he_k$ for $i,j \in \mathbb{Z}$.

Remark 4. *We suppose that $\Pi_h^{(\frac{1}{2},0),(1)}v_h$ is defined in the whole $\bar{\Omega}$. For that purpose we extend the values of v_h by defining that*

$$
\begin{aligned}
v_h(x_{-1/2,j}) &\equiv 2v_h(x_{1/2,j}) - v_h(x_{3/2,j}), \\
v_h(x_{N_1+1/2,j}) &\equiv 2v_h(x_{N_1-1/2,j}) - v_h(x_{N_1-3/2,j}) \qquad (j=0,\cdots,N_2).
\end{aligned}
$$

Similar extension is done for functions operated by the bilinear interpolation $\Pi_h^{(0,\frac{1}{2}),(1)}$.

For this scheme we have the L^2 stability and convergence result. We prepare the following norms and seminorms. For a lattice function ψ_h defined on Ω_h or a set of the functions $\psi_h = \{\psi_h^n\}_{n=0}^{N_T}$,

$$
\begin{aligned}
&\|\psi_h\|_{l^2(\Omega_h)} \equiv \Big\{h^2 \sum_{x_{i,j}\in\Omega_h} \psi_h(x_{i,j})^2\Big\}^{1/2}, \\
&\|\psi_h\|_{\ell^\infty(\ell^2)} \equiv \max_{0\le n\le N_T} \|\psi_h^n\|_{l^2(\Omega_h)}, \quad \|\psi_h\|_{\ell^2(\ell^2)} \equiv \Big\{\Delta t \sum_{n=0}^{N_T} \|\psi_h\|_{l^2(\Omega_h)}^2\Big\}^{1/2}, \\
&|\psi_h|_{h^1(\Omega_h)} \equiv \|\nabla_h\psi_h\|_{l^2(\Omega_h^{(\frac{1}{2},0)})\times l^2(\Omega_h^{(0,\frac{1}{2})})}, \\
&|\psi_h|_{l^2(h^{1\prime})} \equiv \Big\{\Delta t \sum_{n=1}^{N_T} \Big\|\frac{\nabla_h\psi_h^n + \tilde{\nabla}_h^{(n)}\psi_h^{n-1}}{2}\Big\|_{l^2(\Omega_h^{(\frac{1}{2},0)})\times l^2(\Omega_h^{(0,\frac{1}{2})})}^2\Big\}^{1/2},
\end{aligned}
$$

where $\ell^2(\Omega_h^{\frac{1}{2}e_k})$, $k = 1, 2$, is a norm for functions ψ_h defined on the lattice $\bar{\Omega}_h \pm \frac{1}{2}e_k$,

$$\|\psi_h\|_{l^2(\Omega_h^{\frac{1}{2}e_k})} \equiv \left[h^2 \sum\{\psi_h(y_{i,j})^2;\ y_{i,j} = x_{i,j} + \frac{1}{2}e_k \in \bar{\Omega},\ x_{i,j} \in \Omega_h\}\right]^{1/2}.$$

Lemma 3. *Under Hypotheses 1 and 2, scheme* (18) *is L^2-stable, i.e., it holds*

$$\|\phi_h\|_{l^\infty(l^2)} + \sqrt{\nu}|\phi_h|_{l^2(h^{1\prime})} \le c\,(\|\phi^0\|_{l^2(\Omega_h)} + \sqrt{\nu\Delta t}|\phi^0|_{h^1(\Omega_h)} + \|f\|_{l^2(l^2)}).$$

Theorem 3. *Assume hypotheses 1 and 2. Let ϕ and ϕ_h be the solution of* (1) *and* (18)*, respectively. If ϕ, u and f satisfy*

$$\phi \in Z_C^3, \quad u \in Z_C^2, \quad f \in Z_C^2$$

then we have

$$\|\phi - \phi_h\|_{l^\infty(l^2)} + \sqrt{\nu}|\phi - \phi_h|_{l^2(h^{1\prime})} \le c(\|u\|_{Z_C^2})\,(\|\phi\|_{Z_C^3} + \|f\|_{Z_C^2})(h + \Delta t^2).$$

For the proofs of these results we refer to [4], where a scheme having a better convergence order in space is also presented.

6 Conclusion

We have discussed two ways to avoid quadrature in treating characteristic methods for convection diffusion problems from the recent results. One is to use finite difference methods, which does not need quadrature in nature with the loss of geomtrical flexibility. On rectangle domains we have shown two characteristics finite difference schemes of first and second orders in time increment. In the latter case discrete L^2 norm is employed, which will be applicable to the Navier-Stokes equations. The other way is to use the lumping technique for the Galerkin-characteristics finite element method. It keeps the geometrical flexibility of the finite element method, but the convergence order is of first order. To develop numerical schemes having both advantages of these methods is the future work.

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