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Approximate Solutions of Two-Dimensional Caputo Fractional Diffusion Equations

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Abstract

A standard model for non-local diffusive transport, applicable when the heterogeneity length scales are power-law distributed, is to represent the flux in terms of a fractional derivative. Here, a recently proposed scheme for fractional diffusive transport, the control volume weighted flux scheme (CVWFS), which is based on Caputo fractional derivatives, is extended to operate in two or more dimensions. The essential feature in the CVWFS is the representation of the flux at a point as a weighted sum of gradients operating up- and down-stream of that point. Following presentation of the scheme, the convergence and accuracy of the CVWFS, using alternative weightings, is demonstrated and its accuracy illustrated by comparing numerical predictions with two-dimensional analytical solutions.

Keywords: Caputo derivative, fractional diffusion, two-dimensions.

1 Introduction

In a diffusion process an initial pulse will spread with a length scale $\ell \sim t^{\frac{1}{2}}$. In some processes, however, due to the presence of heterogeneities, the time exponent for the spreading length scale can differ from the value of $n = \frac{1}{2}$. Such a process is referred to as anomalous diffusion, with exponent $1 > n > \frac{1}{2}$ called super-diffusion and $n < \frac{1}{2}$ called sub-diffusion [1,2]. Anomalous diffusion can been seen in a number of physical systems including solute transport in a porous media [3,4,5,6,7], earth-surface sediment transport [8,9,10,11], stream solute transport [12], heat transfer in porous media[13], moving boundary problems in heat transfer [2,14] and drug release[15].

In conventional diffusion processes, the flux at a point is proportional to the local gradient of the potential ϕ with the x-component given by

$$q_x = -\nu \frac{\partial \phi}{\partial x} \tag{1}$$

where ν is the suitably dimensioned diffusivity coefficient. In contrast, cases where the length scales of the heterogeneities in the system have a power-law distribution the flux is controlled by non-local properties; a situation that leads to anomalous super-diffusion. A number of theoretical results [2,8,15,16] indicate that this non-local process can be effectively modeled in terms of derivatives of fractional order $0 < \alpha = \frac{1-n}{n} \leq 1$. One such realization of this model is to express the flux as a combination of left and right-sided Caputo fractional derivatives

$$q_x = -\nu \frac{1+\beta_x}{2} \frac{\partial^{\alpha_x} \phi}{\partial x^{\alpha_x}} + \nu \frac{1-\beta_x}{2} \frac{\partial^{\alpha_x} \phi}{\partial (-x)^{\alpha_x}} = \frac{1+\beta_x}{2} q_x^L + \frac{1-\beta_x}{2} q_x^R \tag{2}$$

where $-1 \leq \beta_x \leq 1$ is a bias weighting between the left and the right-sided derivatives and $0 < \alpha_x \leq 1$ is the order of the derivative—which can also be viewed as a measure of the non-locality in the system. The left sided (*L*) and right sided (*R*) derivatives in Equation (2) are respectively defined by [17]

$$\frac{\partial^\alpha \phi}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha} \frac{\partial \phi(\xi)}{\partial x} d\xi \tag{3a}$$

$$\frac{\partial^\alpha \phi}{\partial (-x)^\alpha} = \frac{-1}{\Gamma(1-\alpha)} \int_x^1 (\xi - x)^{-\alpha} \frac{\partial \phi(\xi)}{\partial x} d\xi \tag{3b}$$

where Γ is the gamma function. Note that the Caputo definition in Equation (2) and Equation (3) is preferred over the alternative Reimann-Liouville definition because conditions at the boundaries can be more easily constructed to match physically meaningful conditions and the Caputo fractional derivative of a constant is zero [17].

Under the assumption that the heterogeneities in the coordinate directions are statistically independent, see discussions by Tadjeran and Meerschaert [18], and that the domain is scaled so $0 \leq x \leq 1$; $0 \leq y \leq 1$ a general fractional diffusion equation in two dimensions can be written as

$$\frac{\partial \phi}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + S \tag{4}$$

where *S* is a source term, and the definition of q_y is simply obtained by replacing *y* for *x* in Equation (2). In terms of finding approximate solutions of the form in Equation (4), previous research has been based on the so called one-shift Grünwald approximation [8,12,17,18,19,20,21,22] or the alternative L1/L2 approximation [1,6,7,12,22,23,24] for fractional derivatives. Alternatively, Ervin and Roop [25,26] presented the theoretical framework for a Galerkin finite element approximation[25,26] for steady state fractional advection dispersion equations. In particular, Meerschaert and Tadjeran used the one-shift Grünwald approximation with a fully implicit ADI [10] and Crank-Nicolson ADI [18] time stepping schemes and Yang et. al. [27] used the L1/L2 approximation in both finite element and finite difference solutions.

Recently Voller et. al. [28] obtained a solution to the one-dimensional form of Equation (4) using a discrete control volume approach – called the Control Volume Weighted Flux Scheme (CVWFS). The novelty of the approach lies in modeling the local diffusion flux at local control volume face as the weighted average of gradients across the control volume faces up- and down-stream, and creates a physical analog for the non-local diffusion that can be implemented directly into a control volume discretization. Here, we will expand the CVWFS to find numerical solutions for the two-dimensional transient Caputo fractional diffusion problem in Equation (4). Where possible this approach will be verified by using available analytical solutions.

The paper is laid out as follows. In the next section a brief overview of the CVWFS approach applied to a one-dimensional version of Equation (4) is provided and possible alternative schemes for choosing the gradient weights are provided with associated error analysis. Following, the CVWFS approach is generalized to handle the full two-dimensional form of Equation (4). The work concludes with a number of test problems that demonstrate the relative accuracy of the CVWFS when compared to the one-shift Grünwald [8,12,17,18,19,20,21,22] and L1/L2 approximations [1,6,7,12,22,23,24], and verify the proposed two-dimensional CVWFS on solving problems with known analytical solutions.

2 Overview of the CVWFS

Before demonstrating the discretization of the CVWFS in two-dimensions, it is important to understand the main characteristics of the method in one-dimension. The CVWFS is a discrete control volume (or finite volume) method that was developed to numerically solve non-local diffusion transport problems in the scaled domain $0 \le x \le 1$, [28]. In this approach a uniformly spaced grid of $NX+1$ nodes a distance $\Delta x = 1/NX$ apart is assumed—note this will place nodes on the boundary (see Figure 1).

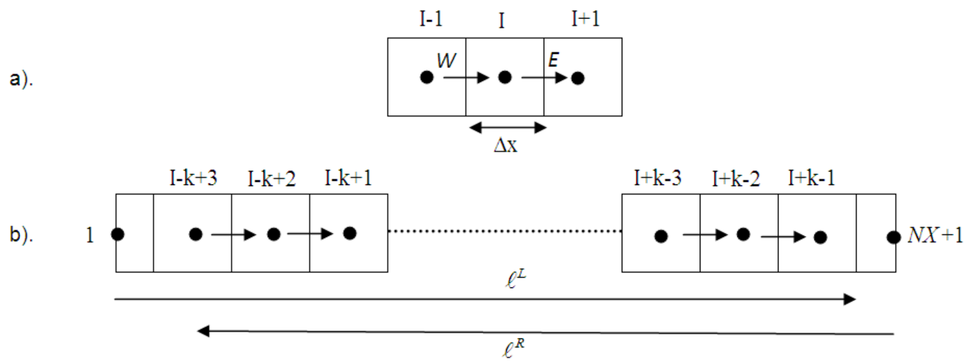

Figure 1: Grids for calculating diffusion flux in one-dimension with a). local diffusion and b). non-local diffusion (Figure modified from Voller et al. Figure 1 [28]). Fluxes from the W-west and E-east are noted at node *I,J*.

In a standard approach the flux into the control volume about node I from the left, defined as q_{west_I}, would be estimated in terms of the potential gradient at the control volume face located at $x = x_I - \Delta x/2$. In a similar manner the flux out to the right, q_{east_I}, would be determined solely by the potential gradient at the face located at $x = x_I + \Delta x/2$. By contrast, in the CVWFS these fluxes are calculated in terms of a weighted average of the potential gradients across multiple control volume faces up- and down-stream of node I (see Figure 1). There are two end-members in this treatment. In the first the flux on a given face of the control volume is calculated in terms of the potential gradients at and on all the neighboring control volume faces positioned to the *Right* of the given face. In this way the flux at the *east* and *west* faces of the I^{th} control volume are written as the weighted average

$$\begin{aligned} q^R_{west_I} &= \frac{\nu_{west_I}}{\Gamma(2-\alpha)} \sum_{k=1}^{NX+2-I} W_k \left[\frac{\phi_{I+k-2} - \phi_{I+k-1}}{\Delta x} \right] \\ q^R_{east_I} &= \frac{\nu_{east_I}}{\Gamma(2-\alpha)} \sum_{k=1}^{NX+1-I} W_k \left[\frac{\phi_{I+k-1} - \phi_{I+k}}{\Delta x} \right] \end{aligned} \tag{5}$$

where W_k are appropriately selected flux weights. Note, in Equation (5), only faces that lie within the domain $0 \le x \le 1$ are included in the weighted sum and that the upper limit of the summation differ by one; a device, as detailed in Voller et. al. [28], that leads to desirable numerical properties. The opposite of Equation (5) is to write the face fluxes as a weighted sum of gradients at the faces that lie to the *Left* of the given face, i.e.

$$\begin{aligned} q^L_{west_I} &= \frac{\nu_{west_I}}{\Gamma(2-\alpha)} \sum_{k=1}^{I-1} W_k \left[\frac{\phi_{I-k} - \phi_{I+1-k}}{\Delta x} \right] \\ q^L_{east_I} &= \frac{\nu_{east_I}}{\Gamma(2-\alpha)} \sum_{k=1}^{I} W_k \left[\frac{\phi_{I+1-k} - \phi_{I+2-k}}{\Delta x} \right] \end{aligned} \tag{6}$$

In applying Equations (5) and (6) to solve fractional diffusion problems, Voller et. al. [28] proposed using the power law weights

$$\begin{aligned} W_k &= (1-\alpha)[(k-\mu)]^{-\alpha} \Delta x^{1-\alpha} \\ 0 < \mu &= \left[1 - \Gamma(1-\alpha)^{-\frac{1}{\alpha}} \right] < 1 \end{aligned} \tag{7}$$

The following comments are made:

1. As shown in [28], a connection to the Caputo derivate definition in Equation (3) is established by noting that the components in Equations (5) and (6) are

respectively formal approximations of q_x^R and q_x^L defined in Equation (2) when a constant of $\frac{1}{(1-\alpha)\Gamma(1-\alpha)} = \frac{1}{\Gamma(2-\alpha)}$ is multiplied by the weights generated by Equation (7).

2. The form of the correction μ allows for the recovery of the correct local approximation for the flux as $\alpha \to 1$.

3. A discrete flux balance over a control volume, using Equations (5) and/or (6), will lead to a system of discrete equations that, in a steady state case, will be diagonally dominant.

3 Alternative Weights

The power-law weights in Equation (7) can be replaced by alternative weighting schemes derived from other fractional derivative algorithms developed in the literature. One example is the L1/L2 algorithm for approximation of Caputo derivatives [1,6,12,22,24,26,29,30]. When applied to the CVWFS framework the L1/L2 weights are given by

$$W_k^{L1/L2} = \left[k^{1-\alpha} - (k-1)^{1-\alpha}\right]\Delta x^{1-\alpha}, \quad 1 \le k \le NX \tag{8}$$

Another example are weights obtained from the classical Grünwald approximation for a fractional derivative [8,12,17,21], which can be manipulated to the CVWFS form in Equations (5) and (6) to provide the following alternative definition for the weights in Equation (7)

$$W_k^G = \Gamma(2-\alpha)\sum_{i=1}^{k} g_i \Delta x^{1-\alpha} \quad 1 \le k \le NX \tag{9}$$

where

$$\begin{aligned} g_1 &= 1 \\ g_i &= \frac{i-2-\alpha}{i-1} g_{i-1}, \quad i = 2,3,\ldots k \end{aligned} \tag{10}$$

Note when these weights are used in the CVWFS the resulting scheme will match the so called one-shift Grünwald schemes for fractional diffusion previously reported in the literature [8,12,17,18,19,20,21,22,28].

Table 1 compares values of $\Delta x^{\alpha-1} W_k / \Gamma(2-a)$ (k=1 .. 5) for the case $\alpha = 0.3$, using all three methods. Observe the close match between the CVWFS weights (Equation (7)) and the Grünwald weights (Equation (9)) for all k=1,2… and the initial differences in the L1/L2 weights (Equation (8)). In respect to the first observation, it is noted that close agreement between the Grünwald and CVWFS weights is achieved through the choice of the correction factor μ in Equation (7); which enforces an exact match for the first weight, k=1.

k	CVWFS	Grünwald	L1/L2
1	1.0000	1.0000	1.1005
2	0.6936	0.7000	0.6873
3	0.5910	0.5950	0.5868
4	0.5328	0.5355	0.5297
5	0.4933	0.4953	0.4910

Table 1: Alternative normalized weights for CVWFS

4 Extension of CVWFS to 2D Caputo Fractional Diffusion Equations

The approximate solution of the 2-D fractional diffusion problem in Equation (4) requires a straight forward extension of the 1-D CVWFS [28] outlined above. The key idea is to treat the *y* derivative approximations to the *north* (above) and *south* (below) of node *I,J* in the same fashion as the left and right hand *x* derivative approximations defined by Equations (5) and (6). The starting point is to assume a scaled domain of $(NX+1)\times(NY+1)$ nodes, where node *I,J* is located on the I^{th} column and J^{th} row of the discretization displayed in Figure 2.

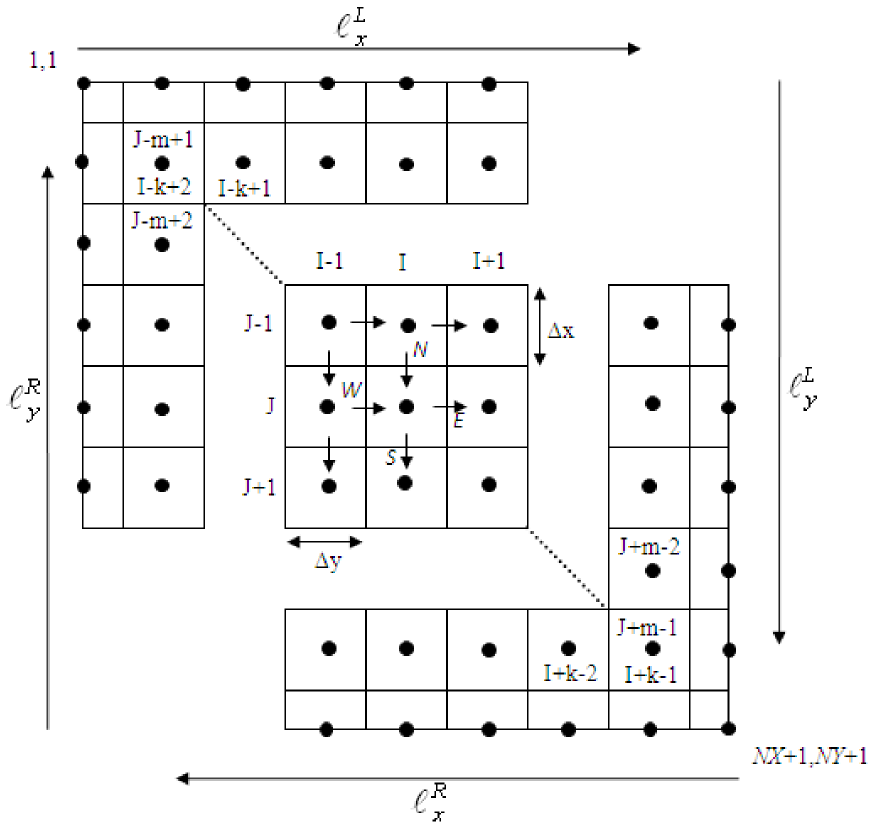

Figure 2: Grid for calculating diffusion flux in two-dimensions. Subscripts on ℓ indicate direction of non-localities. Fluxes from N-north, S-south, E-east, W-west are noted at node *I,J*.

Then, following directly from Equations (5) and (6), the fluxes in the x-direction on the *east* and *west* faces of the control volume around node *I,J* can be written as the weighted averages

$$\begin{aligned} q^R_{west_{I,J}} &= \frac{v_{west_{I,J}}}{\Gamma(2-\alpha_x)} \sum_{k=1}^{NX+2-I} W^x_k \left[\frac{\phi_{I+k-2,J} - \phi_{I+k-1,J}}{\Delta x} \right] \\ q^R_{east_{I,J}} &= \frac{v_{east_{I,J}}}{\Gamma(2-\alpha_x)} \sum_{k=1}^{NX+1-I} W^x_k \left[\frac{\phi_{I+k-1,J} - \phi_{I+k,J}}{\Delta x} \right] \end{aligned} \tag{11}$$

$$\begin{aligned} q^L_{west_{I,J}} &= \frac{v_{west_{I,J}}}{\Gamma(2-\alpha_x)} \sum_{k=1}^{I-1} W^x_k \left[\frac{\phi_{I-k,J} - \phi_{I+1-k,J}}{\Delta x} \right] \\ q^L_{east_{I,J}} &= \frac{v_{east_{I,J}}}{\Gamma(2-\alpha_x)} \sum_{k=1}^{I} W^x_k \left[\frac{\phi_{I+1-k,J} - \phi_{I+2-k,J}}{\Delta x} \right] \end{aligned} \tag{12}$$

The weighted average of fluxes in the y-direction is restricted to that of gradients across the faces of control volumes along column *I*. The *north* and *south* flux at node *I,J* based on the derivatives from the left and right of the control volume can then be written as:

$$\begin{aligned} q^R_{north_{I,J}} &= \frac{v_{north_{I,J}}}{\Gamma(2-\alpha_y)} \sum_{m=1}^{NY+2-J} W^y_m \left[\frac{\phi_{I,J+m-2} - \phi_{I,J+m-1}}{\Delta y} \right] \\ q^R_{south_{I,J}} &= \frac{v_{south_{I,J}}}{\Gamma(2-\alpha_y)} \sum_{m=1}^{NY+1-J} W^y_m \left[\frac{\phi_{I,J+m-1} - \phi_{I,J+m}}{\Delta y} \right] \end{aligned} \tag{13}$$

$$\begin{aligned} q^L_{north_{I,J}} &= \frac{v_{north_{I,J}}}{\Gamma(2-\alpha_y)} \sum_{m=1}^{J-1} W^y_m \left[\frac{\phi_{I,J-m} - \phi_{I,J+1-m}}{\Delta y} \right] \\ q^L_{south_{I,J}} &= \frac{v_{south_{I,J}}}{\Gamma(2-\alpha_y)} \sum_{m=1}^{J} W^y_m \left[\frac{\phi_{I,J+1-m} - \phi_{I,J+2-m}}{\Delta y} \right] \end{aligned} \tag{14}$$

Here, $NX\Delta x = NY\Delta y$ and W^x_k and W^y_m are the flux weight at each control volume face defined by Equation (7) using α_x and α_y, respectively.

Using the approximations of Equation (11-14) the components of divergence of the fractional diffusion on the right hand side of Equation (4) can be approximated as

$$\begin{aligned} -\left.\frac{\partial q_x}{\partial x}\right|_{I,J} &\approx \frac{1+\beta_x}{2} \frac{q^L_{west_{I,J}} - q^L_{east_{I,J}}}{\Delta x} + \frac{1-\beta_x}{2} \frac{q^R_{west_{I,J}} - q^R_{east_{I,J}}}{\Delta x} \\ -\left.\frac{\partial q_y}{\partial y}\right|_{I,J} &\approx \frac{1+\beta_y}{2} \frac{q^L_{north_{I,J}} - q^L_{south_{I,J}}}{\Delta y} + \frac{1-\beta_y}{2} \frac{q^R_{north_{I,J}} - q^R_{south_{I,J}}}{\Delta y} \end{aligned} \tag{15}$$

Hence, on using an explicit approximation in time, the following first order accurate in time and second order accurate in space scheme can be generated for the governing Equation (4)

$$\phi_{I,J}{}^{new} = \phi_{I,J} + \Delta t\left[-\left.\frac{\partial q_x}{\partial x}\right|_{I,J} - \left.\frac{\partial q_y}{\partial y}\right|_{I,J} \right] + \Delta t S_{I,J} \tag{16}$$

where *new* denotes the new temperature at node *I,J* at time $t + \Delta t$. Note, a steady state solution of Equation (4) would also be based on Equation (16). In this case, however, *new* would designate a pseudo-time step in an iterative solution.

As noted in [28] the CVWFS can naturally handle fixed value boundary conditions by simple substitution of the given values at the extrema of the summations in Equations (11) and (14). An appropriate substitution is also made to account for the prescription of a fixed flux condition. For example if a fixed flux q is applied on the right boundary x = 1, an iterative update of the boundary values is made to ensure a correct boundary flux calculation, see [28]

$$-\left.\frac{\partial q_x}{\partial x}\right|_{NX+I,J} = \frac{1+\beta_x}{2}\frac{q^L_{west_{NX+1,J}} - q}{0.5\Delta x} + \frac{1-\beta_x}{2}\frac{q^R_{west_{NX+1,J}} - q}{0.5\Delta x} \tag{17}$$

where the factor of 0.5 in the denominator accounts for the half sized control volume at the boundary.

While Equation (16) is the preferred form for computations it is worthwhile for analysis to expand Equation (16) using Equations (11-15) to arrive at the following point scheme

$$\phi_{I,J}{}^{new} = a_{I,J}\phi_{I,J} + \sum_{k=1,k\neq I}^{NX+1} a_{k,J}\phi_{k,J} + \sum_{m=1,m\neq J}^{NY+1} a_{I,m}\phi_{I,m} + \Delta t S_{I,J} \tag{18}$$

where the a's are coefficients as defined in [28] for one-dimension. In Equation (18) it is noted, assuming a diffusivity of 1.0 for notational convenience, that the only possible negative coefficient is

$$a_{I,J} = 1 + \Delta t\left[\frac{\Delta x^{\alpha_x - 1}}{\Gamma(2-\alpha_x)\Delta x^{1+\alpha_x}}\left(-2W_1^x + W_2^x\right) + \frac{\Delta y^{\alpha_y - 1}}{\Gamma(2-\alpha_y)\Delta y^{1+\alpha_y}}\left(-2W_1^y + W_2^y\right) \right] \tag{19}$$

Consistent with known behaviors in numerical solutions of normal diffusion, Equation (18) shall provide a stable solution provided all the coefficients on the right-hand side are positive. Therefore, setting $a_{I,J} \geq 0$ allows for derivation of stability criteria. In this way using the observation from Table 1 that, for all α's $W_1 = \Gamma(2-\alpha)\Delta x^{1-\alpha}$ and $W_2 \approx (1-\alpha)\Gamma(2-\alpha)\Delta x^{1-\alpha}$ a stability criterion for Equation (16) sets the time step as

$$\Delta t < \left[\frac{1+\alpha_x}{\Delta x^{\alpha_x+1}} + \frac{1+\alpha_y}{\Delta y^{\alpha_y+1}} \right]^{-1} \tag{20}$$

consistent with stability criteria for the one-dimensional CVWFS [28] and other explicit fractional transient diffusion schemes in the literature [19].

5 Testing and Results

5.1 Comparison of Weighting Schemes

As noted above with the proposed CVWFS three alternative weighting schemes can be used, the original weights suggested by Voller et al [28] (Equation (7)), weights based on the L1/L2 algorithm (Equation (8)) and weights derived from the Grünwald approximation (Equation (9)). To provide an illustration on the relative accuracy of these three alternatives we consider the following one-dimensional Caputo fractional diffusion equation

$$\frac{\partial}{\partial x}\left(\frac{\partial^\alpha \phi}{\partial x^\alpha} \right) = 0, \; 0 \le x \le 1 \tag{21}$$

With

$$\phi(0) = 1, \text{ and } \phi(1) = 0 \tag{22}$$

The analytical solution to Equation (21) with Equation (22) is

$$T = 1 - x^\alpha \tag{23}$$

The accuracy of each respective flux weight system is analyzed by inserting each weight into the 1-D steady state problem in the x-direction of Equation (16) using the pseudo time stepping CVWFS approach. The L-infinity norm—maximum absolute error—was calculated for each weighting scheme for varying grid sizes and α 's (see Table 2).

As might be expected from the close match between the weights in Table 1, similar values of the relative error between the Grünwald and CVWFS weights is observed, with both methods being significantly more accurate than predictions obtained with the L1/L2 weights. Based on the above tests, the remaining applications of the CVWFS will all use the original weights suggested by Voller et al [28] given in Equation (7) and will be implemented on grids of size $\Delta x, \Delta y \le 0.00625$.

Δx	CVWFS	Grünwald	L1/L2
	$\alpha = 0.3$		
0.10000	0.04820	0.04661	0.08591
0.05000	0.04070	0.03981	0.07253
0.02500	0.03361	0.03312	0.05998
0.01250	0.02748	0.02722	0.04914
0.00625	0.02237	0.02224	0.04009
	$\alpha = 0.7$		
0.10000	0.01762	0.01633	0.03212
0.05000	0.01137	0.01064	0.02117
0.02500	0.00713	0.00673	0.01348
0.01250	0.00441	0.00420	0.00844
0.00625	0.00271	0.00260	0.00524

Table 2: L-infinity norm for CVWFS approximate solution to the one-dimensional fractional diffusion problem in Equation (25) utilizing each alternative weights in Equations (7)-(9)

5.2 2D Steady State Problems

The initial test problem for the two-dimensional CVWFS is a steady state problem with constant diffusivity $\nu = 1$ and fixed boundary conditions. The general Caputo fractional diffusion equation for this problem is Equation (4) with the transient and source term neglected, i.e.

$$\begin{aligned} &\frac{\partial}{\partial x}\left(-q_x\right)+\frac{\partial}{\partial y}\left(-q_y\right)=0 \\ &q_x = -\frac{1+\beta_x}{2}\frac{\partial^{\alpha_x}\phi}{\partial x^{\alpha_x}}+\frac{1-\beta_x}{2}\frac{\partial^{\alpha_x}\phi}{\partial(-x)^{\alpha_x}} \\ &q_y = -\frac{1+\beta_y}{2}\frac{\partial^{\alpha_y}\phi}{\partial x^{\alpha_y}}+\frac{1-\beta_y}{2}\frac{\partial^{\alpha_y}\phi}{\partial(-x)^{\alpha_y}} \end{aligned} \tag{24}$$

Two alternative boundary conditions are considered. The first sets

$$\begin{aligned} &\phi(x,0) = 1 - x^{\alpha_x} \\ &\phi(0,y) = 1 - y^{\alpha_y} \\ &\phi(x,1) = \phi(1,y) = 0 \end{aligned} \tag{25}$$

The second sets

$$\begin{aligned} &\phi(x,0) = (1 - x)^{\alpha_x} \\ &\phi(0,y) = (1 - y)^{\alpha_y} \\ &\phi(x,1) = \phi(1,y) = 0 \end{aligned} \tag{26}$$

In the limit case of $\beta_x = \beta_y = 1$ (only selecting the left-hand fractional derivatives) the solution to Equations (24) and (25) is

$$\phi(x,y) = \left(1 - x^{\alpha_x}\right)\left(1 - y^{\alpha_y}\right) \tag{27}$$

In the alternative limit case of $\beta_x = \beta_y = -1$ (only selecting the right-hand derivatives) the solution to Equations (24) and (26) is

$$\phi(x,y) = (1-x)^{\alpha_x}(1-y)^{\alpha_y} \tag{28}$$

Solutions that may be verified by direct differentiation and substitution in Equation (24) using the following standard Caputo fractional derivatives relationships

$$\frac{\partial^\alpha}{\partial x^\alpha} A x^\alpha = A\Gamma(\alpha+1), \quad A = f(y) \tag{29}$$

$$\frac{\partial^\alpha}{\partial(-x)^\alpha} f(x) \equiv \frac{\partial^\alpha}{\partial x^\alpha} f(1-x); \quad x \subset [0,1] \tag{30}$$

The CVWFS solutions of the above problems are achieved using the approximations of Equations (11)-(15) in the pseudo time step iterative solver of Equation (16). The CVWFS solution to each set of boundary conditions was solved using $\alpha_x = \alpha_y = 0.3$, $\beta_x = \beta_y = 1$ and $\alpha_x = \alpha_y = 0.7$, $\beta_x = \beta_y = -1$; a grid spacing of $\Delta x = \Delta y = 0.025$ and a pseudo time $\Delta t = 0.0005$—sufficient for stability in both cases—were also selected.

Table 3 provides the maximum absolute error for each problem using all three weighting schemes. Again, the CVWFS and Grünwald weights are in close agreement. Figure 3 displays the CVWFS values along the line of symmetry (x=0, y=0 to x=1, y =1) with the available analytical solutions. The results in Figure 3 are sufficient to verify that the proposed CVWFS can produce accurate solutions to two-dimensional fractional diffusion equations utilizing the weights provided by Equations (7)-(9).

Equations	CVWFS	Grünwald	L1/L2
(24),(25),(27)	0.0294	0.0282	0.0512
(24),(26),(28)	0.0057	0.0053	0.0105

Table 3: Maximum absolute error of CVWFS approximation using alternative weights on grid spacing of $\Delta x = \Delta y = 0.025$

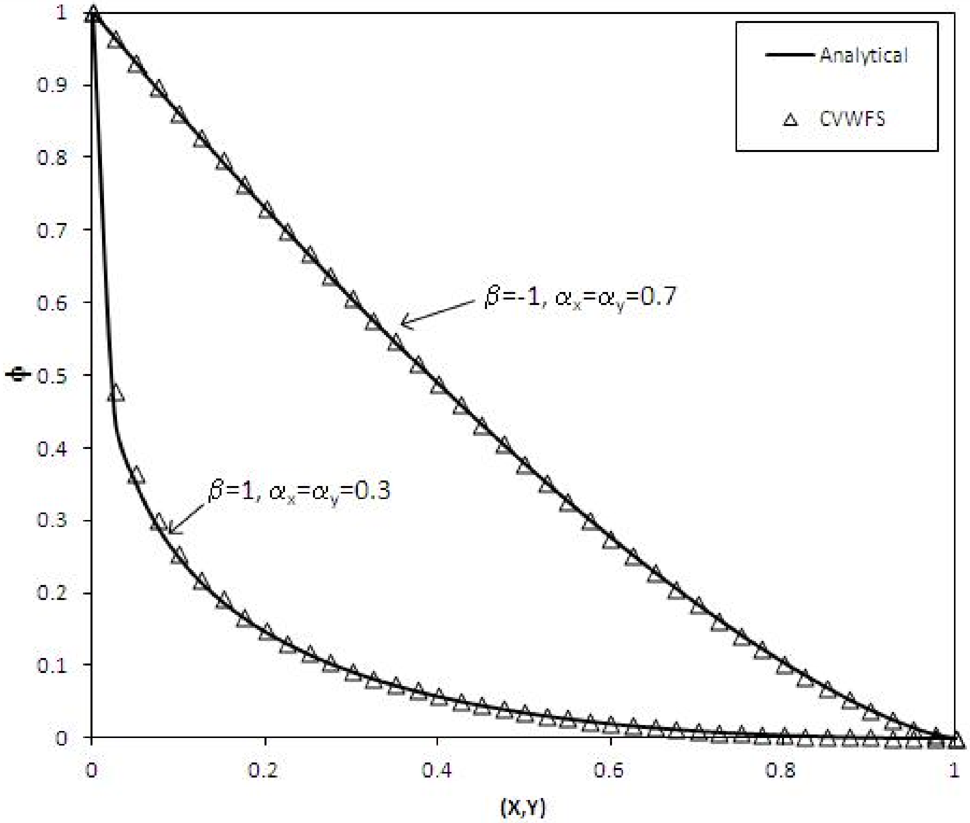

Figure 3: CVWFS and random walk solution to steady state problems. Values presented along the line of symmetry (x=0, y=0 to x=1,y=1).

5.2 A Transient Problem

Similar to work in [18,20] the following left-hand only fractional diffusion transient problem is posed

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(\nu_x \frac{\partial^{\alpha_x} \phi}{\partial x^{\alpha_x}} \right) + \frac{\partial}{\partial y}\left(\nu_y \frac{\partial^{\alpha_y} \phi}{\partial y^{\alpha_y}} \right) + S \tag{31}$$

with boundary and initial conditions

$$\begin{aligned} &\phi(x,0,t) = \phi(0,y,t) = 0 \\ &\phi(1,y,t) = e^{-t} y^{3.6} \\ &\phi(x,1,t) = e^{-t} x^{3} \\ &\phi(x,y,0) = x^{3} y^{3.6} \end{aligned} \tag{32}$$

and a source term set as

$$S(x,y,t) = -(1+2xy)e^{-t}x^3y^{3.6} \tag{33}$$

Setting the diffusivities to

$$\begin{aligned} v_x &= \frac{\Gamma(4-\alpha_x)}{30}x^{2+\alpha_x}y \\ v_y &= \frac{6}{(5+\alpha_y)\Gamma(4+\alpha_y)}y^{2+\alpha_y}x \end{aligned} \tag{34}$$

Leads to the following analytical solution for Equations (31)-(34)

$$\phi(x,y,t) = e^{-t}x^3y^{3.6} \tag{35}$$

Which is readily verified by direct differentiation and substitution in Equation (31) using the following standard Caputo fractional derivative relationship

$$\frac{\partial^\alpha}{\partial x^\alpha}x^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\alpha)}x^{\eta-\alpha} \tag{36}$$

Here in addition to testing the ability of the CWVFS to deal with transient problems its ability to deal with different localities in the coordinate directions is also tested by setting $\alpha_x = 0.8$ and $\alpha_y = 0.6$. In the CVWFS solution (see scheme defined by Equation (16)) the spatial and temporal grid spaces were set at $\Delta x = \Delta y = 0.025$ and $\Delta t = 0.0005$, choices that satisfy the stability criteria of Equation (20). Prediction of the profile along the line of symmetry (x=0, y=0 to x=1, y=1) at time t = 1 are compared with the analytical solution in Figure 4. The maximum absolute error for this grid size is 5.77E-4. The maximum error, at the same time, for a grid size set at $\Delta x = \Delta y = 0.0125$ and $\Delta t = 0.0001$ drops to 4.98E-5. Both maximum errors were found to be consistent with those reported in the literature using alternative solution schemes [18,20].

6 Conclusions

Recent work introduced the so-called control volume weighted flux scheme (CVWFS) for solving one-dimensional steady and transient diffusion equations in which the flux term is expressed in terms of a fractional derivative of order $0 < \alpha \leq 1$ [28]. An approach that can be viewed as an alternative to methods based on the L1/L2 [1,6,7,12,22,23,24] or one-shift Grünwald [8,12,17,18,19,20,21,22] approximations. The main contributions of this paper have been to:

1. Explicitly test the relative accuracy of the previously proposed CVWFS.

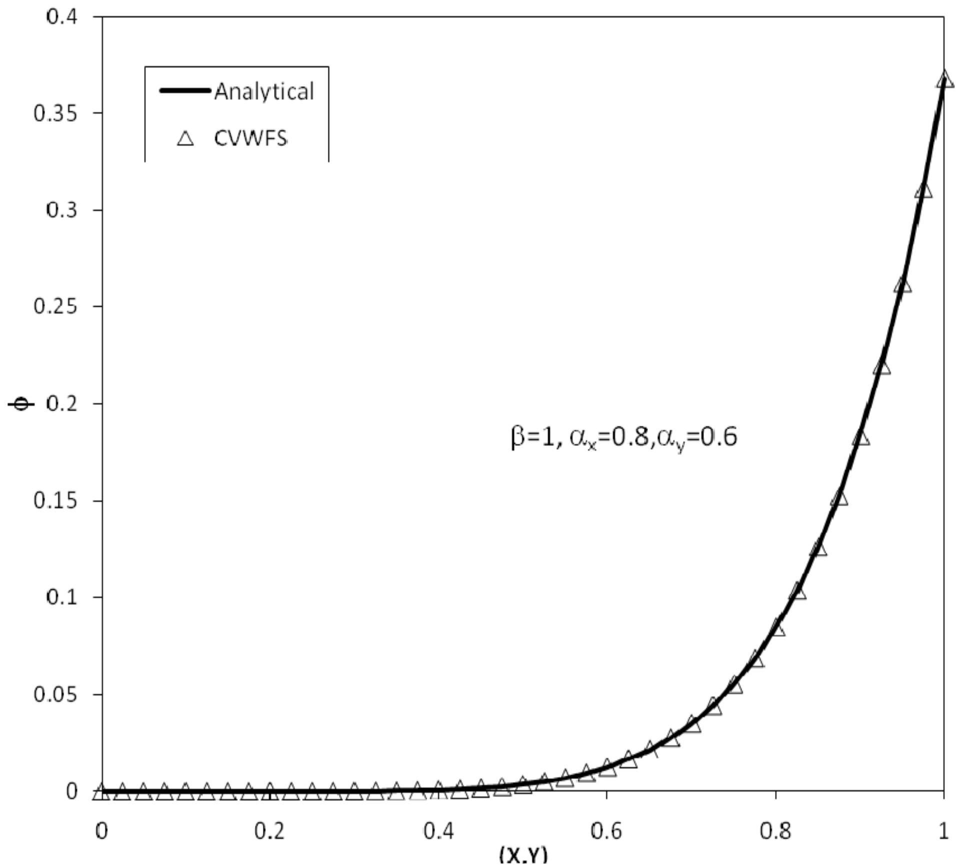

Figure 4: CVWFS and exact solution to transient problem. Values presented along the line of symmetry (x=0, y=1 to x=1, y=1).

2. Demonstrate that the CVWFS can also operate with weights derived from previous literature schemes.
3. Extend and verify the CVWFS for the solution of two-dimensional transient and steady-state fractional diffusion equations.

In the context of a particular one-dimensional test problem, it is clearly demonstrated that the accuracy of the CVWFS using the proposed weights is of the same order as the scheme operating with the Grünwald weights. Further, for a full range of non-locality values α, including different values in the coordinate directions, solutions of the extended two-dimensional CVWFS are in good agreement with available analytical solutions. A multi-dimensional CVWFS can be derived for use on a uniform, structured grid by simply expanding the approximations of equations (11-16) to higher dimensions. Further work will focus on extension of the CVWFS to operate on unstructured finite element grids.

References

[1] V.E. Lynch, B.A. Carreras, D. del-Castillo-Negrete, K.M. Ferreira-Mejias, H.R. Hicks, "Numerical methods for the solution of partial differential equations of fractional order", Journal of Computational Physics, 192, 406-421, 2003.

[2] V.R. Voller, "An exact solution of a limit case Stefan problem governed by a fractional diffusion equation", International Journal of Heat and Mass Transfer, 53, 5622-5625, 2010.

[3] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, "The fractional-order governing equation of Lévy Motion" Water Resources Research, 36, 1413-1423, 2000.

[4] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, "Application of a fractional advection-dispersion equation", Water Resources Research, 36, 1403-1412, 2000.

[5] Z. Yong, D.A. Benson, M.M. Meerschaert, H. Scheffler, "On Using Random Walks to Solve the Space-Fractional Advection-Dispersion Equations", Journal of Statistical Physics, 123, 89-110, 2006.

[6] X. Zhang, J.W. Crawford, L.K. Deeks, M.I. Stutter, A.G. Bengough, I.M. Young, "A mass balance based numerical method for the fractional advection-dispersion equation: Theory and application", Water Resources Research, 41, W07029, 2005.

[7] X. Zhang, M. Lv, J.W. Crawford, I.M. Young, "The impact of boundary on the fractional advection–dispersion equation for solute transport in soil: Defining the fractional dispersive flux with the Caputo derivatives", Advances in Water Resources, 30, 1205-1217, 2007.

[8] R. Schumer, M.M. Meerschaert, B. Baeumer, "Fractional advection-dispersion equations for modeling transport at the Earth surface", Journal of Geophysical Research, 114, F00A07, 2009.

[9] V.R. Voller, C. Paola, "Can anomalous diffusion describe depositional fluvial profiles?", Journal of Geophysical Research, 115, F00A13, 2010.

[10] V. Ganti, M.M. Meerschaert, E. Foufoula-Georgiou, E. Viparelli, G. Parker, "Normal and anomalous diffusion of gravel tracer particles in rivers", Journal of Geophysical Research, 115, F00A12, 2010.

[11] V.R. Voller, V. Ganti, C. Paola, E. Foufoula-Georgiou, "Does the flow of information in a landscape have direction?", Geophysical Research Letters, 39 (1), L01403, 2012.

[12] C. Shen, M.S. Phanikumar, "An efficient space-fractional dispersion approximation for stream solute transport modeling", Advances in Water Resources, 32, 1482-1494, 2009.

[13] R. Scherer, S.L. Kalla, L. Boyadjiev, B. Al-Saqabi, "Numerical treatment of fractional heat equations", Applied Numerical Mathematics, 58, 1212-1223, 2008.

[14] J. Liu, M. Xu, "Some exact solutions to Stefan problems with fractional differential equations", Journal of Mathematical Analysis and Applications, 351, 536-542, 2009.

[15] L. Xicheng, X. Mingyu, W. Showel, “Analytical solutions to the moving boundary problems with space–time-fractional derivatives in drug release devices”, Journal of Physics A: Math. and Theoretical, 40, 12131, 2007.
[16] R. Metzler, J. Klafter, “The random walk's guide to anomalous diffusion: a fractional dynamics approach”, Physics Reports, 339, 1-77, 2000.
[17] I. Podlubny, “Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications”, Academic Press, San Diego, USA, 1999.
[18] C. Tadjeran, M.M. Meerschaert, “A second-order accurate numerical method for the two-dimensional fractional diffusion equation”, Journal of Computational Physics, 220, 813-823, 2007.
[19] M.M. Meerschaert, C. Tadjeran, “Finite difference approximations for two-sided space-fractional partial differential equations”, Applied Numerical Mathematics, 56, 80-90, 2006.
[20] M.M. Meerschaert, H. Scheffler, C. Tadjeran, “Finite difference methods for two-dimensional fractional dispersion equation”, Journal of Computational Physics, 211, 249-261, 2006.
[21] Y. Zhang, D. Benson, M.M. Meerschaert, E.M. LaBolle, “Space-fractional advection-dispersion equations with variable parameters: Diverse formulas, numerical solutions, and application to the Macrodispersion Experiment site data”, Water Resources Research, 43, W05439, 2007.
[22] Q. Yang, “Novel analytical and numerical methods for solving fractional dynamical systems”, PhD Thesis, Queensland University of Technology, 2010.
[23] M. Ilic, F. Liu, I. Turner, V. Anh, “Numerical Approximation of a Fractional-In-Space Diffusion Equation”, Fractional Calculus and Applied Analysis, 8, 323-341, 2005.
[24] K.B. Oldham, J. Spanier, “The fractional calculus”, Academic Press, New York, USA, 1974.
[25] V.J. Ervin, J.P. Roop, “Variational formulation for the stationary fractional advection dispersion equation”, Numerical Methods for Partial Differential Equations, 22, 558-576, 2006.
[26] J.P. Roop, “Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in R^2”, Journal of Computational and Applied Mathematics, 193, 243-268, 2006.
[27] Q. Yang, I. Turner, F. Liu, M. Ilic, “Novel Numerical Methods for Solving the Time-Space Fractional Diffusion Equation in Two Dimensions”, SIAM Journal on Scientific Computing, 33, 1159-1180, 2011.
[28] V.R. Voller, C. Paola, D.P. Zielinski, “The Control Volume Weighted Flux Scheme (CVWFS) for Non-Local Diffusion and Its Relationship to Fractional Calculus”, Numerical Heat Transfer B, 59, 421-441, 2011.
[29] K. Diethelm, N.J. Ford, A.D. Freed, Y. Luchko, “Algorithms for the fractional calculus: A selection of numerical methods”, Computer Methods in Applied Mechanics and Engineering, 194, 743-773, 2005.
[30] K. Diethelm, “Generalized compound quadrature formulae for finite-part integrals”, IMA Journal of Numerical Analysis, 17, 479-493, 1997.